A Fast Tree Algorithm for the Calculation of Electrical Field in 1.5D Streamer Discharge Simulations

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As the initial stage of various electrical discharges, the streamer discharge has drawn great attention. In the streamer discharge simulation, the electric field computation takes more than 90% of the CPU time. In this paper, we propose a fast tree algorithm which help reduce the computational complexity from $O(N^2)$ (typical for traditional direct method) to $O(N \log N)$. A rigorous error estimation shows the relative error of the tree algorithm reduces exponentially fast with respect to the truncation term and can be controlled adaptively. Numerical examples are presented to validate the accuracy and efficiency.

***Index Terms*—tree algorithm, electric field, disc model, streamer discharge.**

I. Introduction

THE streamer is a type of electrical discharge emerging when a strong electric field is applied to an air gap. The streamer discharge has various applications, which makes its simulation draw great attention.

The most time consuming part to simulate the streamer is the calculation of the electric field, which may occupy about 90% CPU time. This paper focuses on the 1.5D model [1], due to its great potential to simulate very long streamers.

For the 1.5-dimensional model, the Possion's equation is solved analytically using the so-called disc method [1]. Assume there is a disc with net charge density $\sigma(x)$, radius r_d, thickness dy, the electric field it generates at point y along the x-axis is:

$$\mathrm{d}E(y) = \begin{cases} \frac{1}{2\varepsilon_0}\sigma(x)(\frac{x-y}{\sqrt{(y-x)^2+r_d^2}}+1)\mathrm{d}x, & x-y<0; \\ \frac{1}{2\varepsilon_0}\sigma(x)(\frac{x-y}{\sqrt{(y-x)^2+r_d^2}}-1)\mathrm{d}x, & x-y\geq 0. \end{cases} \tag{1}$$

To consider the influence of the electrodes to the electric field, the image charges should be taken into account. We consider the image charges whose distance to the electrodes are less than L, where L is the length of the discharge gap. Integrating over the whole domain, the solution of E is

$$\begin{aligned} E(y) &= \frac{1}{2\varepsilon_0}\Bigg[\int_{-L}^{y}\sigma(x)\left(\frac{x-y}{\sqrt{(x-y)^2+r_d^2}}+1\right)\mathrm{d}x \\ &\quad + \int_{y}^{L}\sigma(x)\left(\frac{x-y}{\sqrt{(x-y)^2+r_d^2}}-1\right)\mathrm{d}x\Bigg]. \end{aligned} \tag{2}$$

In this paper, we propose a tree algorithm to accelerate the calculation of the electric field [2].

II. The Tree Algorithm

If we write integral (2) using sufficient high order Gaussian quadrature, and let $q_j := \frac{\omega_j\sigma_j}{2\varepsilon_0}$ where ω_j is the Gaussian quadrature weight, Eq. (2) reduces to

$$E(y) = \Big(\sum_{j=0}^{m} q_j - \sum_{j=m+1}^{n} q_j\Big) + \sum_{j=0}^{n}\frac{q_j(x_j-y)}{\sqrt{(x_j-y)^2+r_d^2}}. \tag{3}$$

Below we will omit the term $\sum_{j=0}^{m} q_j - \sum_{j=m+1}^{n} q_j$ for brevity, but the principle of the tree algorithm remains unchanged.

The fundamental idea of the tree algorithm is that to group charges that lie close as one single source via multipole expansions for the far-field interaction, while the near-field interaction from the neighboring charges are evaluated directly. Assume a cluster of charges $\{q_j\}_{j=0}^n$ located at $\{x_j\}_{j=0}^n$ are gathering around x_c. To calculate the far-field $E(y)$, i.e. $|y-x_c| \gg 0$ and $|y-r_d| \gg 0$, a crude approximation is

$$E(y) = \sum_{j=0}^{n} q_j\Phi(x_j,y) \approx \Big(\sum_{j=0}^{n} q_j\Big)\Phi(x_c,y), \tag{4}$$

with $\Phi(x,y) := \frac{x-y}{\sqrt{(x-y)^2+r_d^2}}$. However, using Taylor expansion, we have

$$\begin{aligned} \Phi(x,y) &= \sum_{k=0}^{\infty}\frac{1}{k!}\Phi^{(k)}(x_c,y)(x-x_c)^k \qquad (5)\\ &= \sum_{k=0}^{p}\frac{1}{k!}\Phi^{(k)}(x_c,y)(x-x_c)^k + R_p(x), \qquad (6) \end{aligned}$$

where $\Phi^{(k)} = \frac{\partial^k Phi}{\partial x^k}$; $p \in \mathbb{N}$; R_p is the residual and $R_p = \sum_{k=p+1}^{\infty}\frac{1}{k!}\Phi^{(k)}(x_c,y)(x-x_c)^k$. Therefore, we have

$$\begin{aligned} E(y) &= \sum_{j=0}^{n} q_j\left(\sum_{k=0}^{\infty}\frac{1}{k!}\Phi^{(k)}(x_c,y)(x_j-x_c)^k\right) \\ &\approx \sum_{k=0}^{p}\Phi^{(k)}(x_c,y)\left(\sum_{j=0}^{n} q_j\frac{(x_j-x_c)^k}{k!}\right). \end{aligned} \tag{7}$$

Let $p=0$, Eq. (7) reduces to the crude approximation Eq. (4). To calculate $E(y)$, one only needs to calculate $\Phi^{(k)}(x_c,y)$.

We now derive a recurrence formula to calculate $\Phi^{(k)}(x,y)$. It is straightforward that

$$\Phi^{(0)}(x,y) = \frac{x-y}{\sqrt{(x-y)^2+r_d^2}}, \Phi^{(1)}(x,y) = \frac{r_d^2}{\left(\sqrt{(x-y)^2+r_d^2}\right)^3},$$

which implies

$$r_d^2\Phi^{(0)}(x,y) = \Phi^{(1)}(x,y)\big((x-y)^3 + r_d^2(x-y)\big). \tag{8}$$

Differentiate Eq. (8) for k-1 times by the general Leibniz's rule, and do some algebraic simplifications,

$$\begin{aligned}(x-y)((x-y)^2+r_d^2)\Phi^{(k)}(x,y) = \\ \big(r_d^2-(k-1)(3(x-y)^2+r_d^2)\big)\Phi^{(k-1)}(x,y) \\ -3(k-1)(k-2)(x-y)\Phi^{(k-2)}(x,y) \\ -(k-1)(k-2)(k-3)\Phi^{(k-3)}(x,y).\end{aligned} \tag{9}$$

Using Eq. (8) and (9), for a fixed y, we may then calculate $\Phi^{(k)}(x_c, y)$ recursively , for $k = 2, 3, ...p$.

III. Error Estimation

Now we give a rigorous error estimation for the evaluation of electric field using Eq. (7). Without loss of generality, we only consider the case $x_c = 0$. Other cases reduce to $x_c = 0$ case after a simple shift, i.e. $x := x - x_c$.

Define a complex function $f(z) := \frac{z-y}{\sqrt{(z-y)^2+r_d^2}}$ with $z \in \mathbb{C}$, which is an analytic function for $|z| < \sqrt{y^2 + r_d^2}$. By Cauchy's integral formula, for any z satisfying $|z| := r \leq R := |y|$,

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi - z}\mathrm{d}\xi, \quad f^{(k)}(0) = \frac{k!}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi^{k+1}}\mathrm{d}\xi, \tag{10}$$

where $i = \sqrt{-1}$, $\Gamma := \{w \in \mathbb{C} || w| = R\}$ is a contour containing the point z. We have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi}\frac{1}{1-\frac{z}{\xi}}\mathrm{d}\xi \\ &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi}\Big(\sum_{k=0}^{p}(\frac{z}{\xi})^k + \frac{(\frac{z}{\xi})^{p+1}}{1-\frac{z}{\xi}}\Big)\mathrm{d}\xi \\ &= \frac{1}{2\pi i}\Big(\sum_{k=0}^{p} z^k \oint_\Gamma \frac{f(\xi)}{\xi^{k+1}}\mathrm{d}\xi + \oint_\Gamma \frac{f(\xi)}{\xi}\frac{(\frac{z}{\xi})^{p+1}}{1-\frac{z}{\xi}}\mathrm{d}\xi\Big) \\ &= \sum_{k=0}^{p}\frac{f^{(k)}(0)}{k!}z^k + \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi}\frac{(\frac{z}{\xi})^{p+1}}{1-\frac{z}{\xi}}\mathrm{d}\xi \qquad (11) \\ &:= \sum_{k=0}^{p}\frac{f^{(k)}(0)}{k!}z^k + R_p. \qquad (12)\end{aligned}$$

Using the fact $|f(\xi)|$ is bounded for $\xi \in \Gamma$, i.e. $|f(\xi)| \leq M$, we have

$$\begin{aligned} |R_p| &= \left|\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi}\frac{(\frac{z}{\xi})^{p+1}}{1-\frac{z}{\xi}}\mathrm{d}\xi\right| \\ &\leq \frac{1}{2\pi}\oint_\Gamma \max\Big(\left|\frac{(\frac{z}{\xi})^{p+1}}{1-\frac{z}{\xi}}\right|\left|\frac{f(\xi)}{\xi}\right|\Big)\mathrm{d}\xi \\ &\leq \max\Big(\left|\frac{(\frac{z}{\xi})^{p+1}}{1-\frac{z}{\xi}}\right|\Big)\max|f(\xi)| \\ &\leq M\frac{R}{R-r}(\frac{r}{R})^{p+1}. \end{aligned} \tag{13}$$

IV. Validation and Efficiency

First we only calculate the far field to validate the Eq. (7). We randomly generate 10000 charges in $[-0.5, 0.5]$, which is around $x = 0$, and set $r_d = 0.1$, then calculate the electric field at $y = 1$ ($r/R = 0.5$) using different numbers of truncation terms, denoted by p. Results in Tab. I show the relative error decays exponentially with respect to p. When p increases by 5, the error reduces about 50-100 times.

TABLE I
Accuracy with different truncation terms p

p	relative error
5	9.43e-5
10	1.17e-6
15	6.40e-8
20	6.48e-10

Then we evaluate both the near field and far field to test the efficiency of the tree algorithm. Different amount of charge sources, with random amount of charge, are randomly placed in $[0, 1]$, the evaluation locations are the same as the source positions. The field generated by the nearing neighboring charges are calculated directly and others are by Eq. (7). The codes are implemented by C++. Results show that the CPU time cost by the algorithm is roughly $O(N \log N)$, which is much faster than the direct summation when N becomes large.

TABLE II
Time cost comparison with different N charges and targets

N	direct sum (s)	tree code (s)
1e4	0.530	0.062
5e4	11.1	0.296
1e5	46.2	0.609
2e5	183.9	1.23

V. Conclusion

We present in this paper a fast tree algorithm to calculate the electric field in 1.5D streamer discharge simulations. The tree algorithm is based on the Taylor's expansion and a recurrence formula to calculate the expansion coefficients is given following the general Leibniz's rule, which makes the coefficient computation in the algorithm very flexible. An error estimation shows that error decays exponentially as the number of truncation terms increases.

Examples are given to validate the error estimation and show the time cost of the total algorithm is $O(N \log N)$ which may reduce the computation time greatly.

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